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A SOLUTION OF HADWIGER'S COVERING PROBLEM FOR ZONOIDS

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For a convex body $M \subset \mathbb{R}^n$ by $b(M)$ the least integer p is denoted, such that there are bodies $M_1, \ldots, M_p$ each of which is homothetic to M with a positive ratio $k<1$ and $M_1 \cup \ldots \cup M_p \supset M$. H. Martini has proved [7] that $b(M) \leq 3\cdot 2^{n-2}$ for every zonotope $M \subset \mathbb{R}^n$, which is not a parallelotope.

In the paper this Martini's result is extended to zonoids. In the proof some notions and facts of real functions theory are used (points of density, approximative continuity).

Let $M \subset \mathbb{R}^n$ be a compact convex set. We denote by $b(M)$ the least natural number p, such that there are sets $M_1, \ldots, M_p$, each of which is homothetic to M with a positive multiple $k < 1$ and $M_1 \cup \ldots \cup M_p \supset M$. It is clear that, if M is an n-dimensional parallelotope, then $b(M) = 2^n$. H. Hadwiger [1] has formulated a hypothesis, which asserts that if a convex set $M \subset \mathbb{R}^n$ is not a parallelotope, then $b(M) < 2^n$. For $n > 2$ problem of Hadwiger is open up to now. An interesting partial result to this hard covering problem was obtained by M. Lassak [2]. In this paper we give a positive solution of Hadwiger's covering problem for a class of convex bodies, which are called *zonoids*.

Definition. A *zonotope* [3], [4] is a convex polytope, which is the vector sum of segments. A *zonoid* [5] is a limit (in the sense of the Hausdorff metric) of a convergent sequence of zonotopes.

In [5] another definition of zonoids is given. Namely, let $\varphi(s)$, $0 \leq s \leq \ell$, be a vector-function of finite variation, such that the parameter s is the length on the curve $x = \varphi(s)$, i.e. the length of the arc with end points $\varphi(0)$ and $\varphi(s)$ is equal to s. We consider an integral (in Lebesque's sense)

$$x(g) = \int_0^\ell g(s)\varphi'(s)\mathrm{d}s, \tag{1}$$

where $g(s)$ is a measurable function, such that $|g(s)| \leq 1/2$ for every $s \in [0,\ell]$. Then the set of all points $x(g)$ is a zonoid. Conversely, each zonoid may be represented in such a form. For other definitions of zonoids see [6], [4].

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Theorem. *For zonoids the conjecture of Hadwiger is true. More exactly, if a zonoid* $Z \subset \mathbb{R}^n$ *is not a parallelotope, then* $b(Z) \leq 3 \cdot 2^{n-2}$.

In Martini's paper [7] an analogous result is established for zonotopes. We shall obtain a proof of the theorem for zonoids with the help of a complicated limit process, which uses a notion of an approximate derivative [8].

First of all we recall that for every compact body M, Hadwiger's covering problem is equivalent to the illumination problem [9], [11], i.e. the minimal number of bodies, which are homothetic to M with a positive multiple $k<1$ and cover M, is equal to the minimal number of directions, which illuminate the boundary of the body M.

Now we sketch a proof of Martini's result. Let M be an n-dimensional zonotope of $\mathbb{R}^n$. Then there are vectors $e_1, \ldots, e_p$ $(p \geq n)$, such that, up to a translation, M is the set of all points

$$x_1 e_1 + \ldots + x_p e_p \qquad (|x_i| \leq 1/2 \text{ for all } i).$$

We may suppose that e_i is not parallel to e_j for $i \neq j$, and that $e_1, \ldots, e_n$ is a basis in $\mathbb{R}^n$. Then $p>n$, since M is not a parallelotope. Let us denote by M_1 the set of all points $x_1e_1+\ldots+x_ne_n+x_{n+1}e_{n+1}$ and by M_2 the set of all points $x_{n+2}e_{n+2}+\ldots+x_pe_p$ $(|x_i| \leq 1/2)$. Then $M=M_1+M_2$ and therefore $b(M) \leq b(M_1)$ (by virtue of Lemma 1 below). Consequently, it is enough to prove that $b(M_1) \leq 3 \cdot 2^{n-2}$.

Let

$$\lambda_1 e_1 + \ldots + \lambda_n e_n + \lambda_{n+1} e_{n+1} = 0 \tag{2}$$

be a nontrivial linear dependence. Then $\lambda_{n+1} \neq 0$ and at least two of the numbers $\lambda_1, \ldots, \lambda_n$ are not equal to 0 (since e_{n+1} is not parallel to e_i for any $i=1,\ldots,n$). We may suppose (replacing, if necessary, e_i by $-e_i$), that all λ_i are nonnegative and

$$\lambda_{n-1} > 0, \qquad \lambda_n > 0, \qquad \lambda_{n+1} > 0. \tag{3}$$

Each vertex q of M_1 has a form

$$q = \frac{1}{2}(\sigma_1 e_1 + \ldots + \sigma_n e_n + \sigma_{n+1} e_{n+1}), \quad \text{where } |\sigma_i| = 1 \text{ for all } i \tag{4}$$

(although not each of the points (4) is a vertex).

Let us fix a combination of signs $\sigma_1 = \pm 1, \ldots, \sigma_{n-1} = \pm 1$. There are, at most 8 vertices of M_1 with this combination of sings (because, in order to get a vertex, it is sufficient to choose $\sigma_{n-1} = \pm 1$, $\sigma_n = \pm 1$, $\sigma_{n+1} = \pm 1$). It is easy to see that all these 8 vertices are illuminated by the following *three* directions

$$a_1 = -f + \varepsilon(e_n - e_{n-1}), \qquad a_2 = -f + \varepsilon(e_{n+1} - e_n), \qquad a_3 = -f + \varepsilon(e_{n-1} - e_{n+1}),$$

where $f = (\sigma_1 e_1 + \ldots + \sigma_{n-2} e_{n-2})/2$ and $\varepsilon > 0$ is small enough. Indeed, according to (2) and (3):

two vertices (4) with $\sigma_{n-1}=1$, $\sigma_n=-1$ are illuminated by a_1;
two vertices (4) with $\sigma_n=1$, $\sigma_{n+1}=-1$ are illuminated by a_2;
two vertices (4) with $\sigma_{n+1}=1$, $\sigma_{n-1}=-1$ are illuminated by a_3;
two vertices (4) with $\sigma_{n-1}=\sigma_n=\sigma_{n+1}$ are illuminated by any a_i.

Thus, all vertices with a fixed combination of signs $\sigma_1, \ldots, \sigma_{n-2}$ are illuminated by three directions. Since there are 2^{n-2} combinations of signs $\sigma_1 = \pm 1, \ldots, \sigma_{n-2} = \pm 1$, then all the vertices of M_1 may be illuminated by $3 \cdot 2^{n-2}$ directions. Consequently, by [9], $b(M_1) \leq 3 \cdot 2^{n-2}$.

We now establish several lemmas, which will give us a way to prove the theorem in the general case.

Lemma 1. *Let a compact convex body $M \subset \mathbb{R}^n$ be a vector sum of two convex sets M_1, M_2. If M_1 is n-dimensional, then $b(M) \leq b(M_1)$.*

Proof. We put $b(M_1) = h$. Let $a_1, \ldots, a_h$ be vectors in $\mathbb{R}^n$, whose directions illuminate the boundary $\operatorname{bd} M_1$ of the convex body M_1. Let c be an arbitrary boundary point of M. Then there are points $c_1 \in M_1$, $c_2 \in M_2$, such that $c = c_1 + c_2$. It is clear that c_1 is a *boundary* point of M_1 (otherwise c could not be a boundary point of M). Let j be an index, such that the point $c_1 \in \operatorname{bd} M_1$ is illuminated by the direction a_j, i.e. there is an interior point $x_1 \in M_1$, such that $x_1 - c_1 = \lambda a_j$, $\lambda > 0$. We put $x = x_1 + c_2$. Then x is an interior point of M (since $x_1 \in \operatorname{int} M_1$, $c_2 \in M_2$). Further,

$$x - c = (x_1 + c_2) - (c_1 + c_2) = x_1 - c_1 = \lambda a_j,$$

and therefore the point $c \in \operatorname{bd} M$ is illuminated by the direction a_j. Thus each boundary point c of M is illuminated by a direction a_j $(j = 1, \ldots, h)$. Consequently, by [9], $b(M) \leq h = b(M_1)$. ∎

The following lemma affirms that the function $b(M)$, defined on the family of all convex sets in $\mathbb{R}^n$, possesses the upper semicontinuity property.

Lemma 2. *If a sequence $M_1, M_2, \ldots$ of compact convex bodies converges to a convex body M, then $b(M) \geq \overline{\lim_{k \to \infty}}\, b(M_k)$.*

Proof. We put $b(M) = h$. Let $a_1, \ldots, a_h$ be vectors in $\mathbb{R}^n$, such that their directions illuminate $\operatorname{bd} M$. Then there are compact sets $\mathcal{D}_1, \ldots, \mathcal{D}_h$, such that $\mathcal{D}_1 \cup \ldots \cup \mathcal{D}_h = \operatorname{bd} M$ and every point $x \in \mathcal{D}_j$ is illuminated by the direction a_j. Let us denote by T_j^λ the translation $x \mapsto x + \lambda a_j$ in $\mathbb{R}^n$. We fix a number $\lambda > 0$, such that $T_j^\lambda(\mathcal{D}_j) \subset \operatorname{int} M$ for every $j = 1, \ldots, h$. Let us denote by V an open set in $\mathbb{R}^n$, such that $\operatorname{cl} V \subset \operatorname{int} M$ and $V \supset T_1^\lambda(\mathcal{D}_1) \cup \ldots \cup T_h^\lambda(\mathcal{D}_h)$. We choose for every $j = 1, \ldots, h$ an open set $W_j \subset \mathbb{R}^n$, such that $W_j \supset \mathcal{D}_j$ and $T_j^\lambda(W_j) \subset V$. Then $W_1 \cup \ldots \cup W_h \supset \operatorname{bd} M$.

Let $\delta > 0$ be such that if $d(M,N) < \delta$ then $\operatorname{cl} V \subset \operatorname{int} N$ and $\operatorname{bd} N \subset W_1 \cup \ldots \cup W_h$ (where d is the Hausdorff metric). We are going to show that if $d(M,N) < \delta$, then $b(N) \leq b(M)$. Indeed, let $x \in \operatorname{bd} N$. Then there is j, such that $x \in W_j$. Consequently,

$$x + \lambda a_j = T_j^\lambda(x) \in T_j^\lambda(W_j) \subset V \subset \operatorname{cl} V \subset \operatorname{int} N,$$

i.e. the point x is illuminated by the direction a_j. Thus, each boundary point x of the body N is illuminated by a direction a_j $(j = 1, \ldots, h)$, i.e. $b(N) \leq h = b(M)$.

Let now $M_1, \ldots, M_2, \ldots$ be a sequence of compact convex bodies in $\mathbb{R}^n$, which converges to a convex body M. There is an integer k_0, such that $d(M, M_k) < \delta$ for every $k > k_0$. Consequently, $b(M_k) \leq b(M)$ for all $k > k_0$ and therefore $\overline{\lim_{k \to \infty}}\, b(M_k) \leq b(M)$. ∎

The following examples illustrate the statement of Lemma 2.

Example 1. Let us introduce Cartesian coordinates $x_1,\dots,x_n$ in $\mathbb{R}^n$ and denote by M_k the parallelotope, which is described by the inequalities

$$|x_1|\le 1,\quad |x_2|\le 1,\quad \dots,\quad |x_{n-1}|\le 1,\quad |x_n|\le 1/k.$$

The parallelotopes M_k become more and more "thin" when $k\to\infty$, and the sequence $M_1, M_2,\dots$ converges to the $(n-1)$-dimensional parallelotope M defined by the relations

$$|x_1|\le 1,\quad |x_2|\le 1,\quad \dots,\quad |x_{n-1}|\le 1,\quad x_n = 0.$$

Thus $b(M_k)=2^n$, $b(M)=2^{n-1}$, i.e. the inequality of Lemma 2 is not true in this case. This means that the requirement in Lemma 2 that M be a body is essential.

Example 2. Let $M\subset\mathbb{R}^n$ be an n-dimensional parallelotope and M_k its $1/k$-neighborhood, $k=1,2,\dots$. Then $\lim\limits_{k\to\infty} M_k=M$ and $b(M_k)=n+1$, $b(M)=2^n$. It means that the inequality $b(M)\le \varliminf\limits_{k\to\infty} b(M_k)$ is not true in this case, i.e. unfortunately, there is no corresponding property lower semicontinuity.

By the way lower semicontinuity would allow to obtain our theorem by a direct limit process (from Martini's result). In this connection we use a suitable indirect method.

Let $Z\subset\mathbb{R}^n$ be a zonoid, $\varphi(s)$, $0\le s\le \ell$, its determining vector-function (cf. (1)) and $s_1,\dots,s_k$ points of segment $[0,\ell]$, such that at each of these points the derivative $\varphi'(s)$ exists and is approximately continuous. Then the vector sum of k segments, respectively parallel to the vectors $\varphi'(s_1),\dots,\varphi'(s_k)$, is called a *tangential zonotope* of zonoid Z.

Lemma 3. *Let $Z\subset\mathbb{R}^n$ be a zonoid and M one of its tangential zonotopes. If M is a body, then $b(Z)\le b(M)$.*

Proof. Let $\varphi(s)$, $0\le s\le\ell$, be a determining vector-function of Z (cf. (1)). Then there are points $s_1,\dots,s_k\in[0,\ell]$, at which $\varphi'(s)$ exists and is approximately continuous, and vectors $e_1,\dots,e_k$, which are correspondingly parallel to $\varphi'(s_1),\dots,\varphi'(s_k)$, such that M is the set of all points $\mu_1e_1+\dots+\mu_ke_k$, where $|\mu_i|\le 1/2$ for all i. We have $\varphi'(s_i)=(1/\lambda_i)\cdot e_i$ where λ_i is the length of the vector e_i, $i=1,\dots,k$.

Let $b(M)=h$ and $a_1,\dots,a_h$ be vectors, such that their directions illuminate the boundary of the convex body M. Now let us construct for M the sets $\mathcal{D}_j$, V, W_j and the numbers λ, δ in the same way as in the proof of Lemma 2.

Since $\varphi'(s)$ is approximately continuous at the point s_i, there is a set $P_i\subset[0,\ell]$, such that s_i is a point of density of the set P_i and at each point $s\in P_i$ the derivative $\varphi'(s)$ exists and satisfies the inequality $|\varphi'(s)-\varphi'(s_i)|<\delta\cdot(\lambda_1+\dots+\lambda_k)^{-1}$. We may suppose that each two of the sets $P_1,\dots,P_k$ are disjoint. Finally (passing to subsets if necessary), we may suppose that $\operatorname{mes}P_i=q\lambda_i$, $i=1,\dots,k$, where q is a positive number.

We put $P=P_1\cup\dots\cup P_k$, $Q=[0,\ell]\setminus P$. Let us denote by Z_1, Z_2 the sets of all points

$$z_1(g)=\int\limits_P g(s)\varphi'(s)\mathrm{d}s \qquad z_2(g)=\int\limits_Q g(s)\varphi'(s)\mathrm{d}s$$

respectively, where $g(s)$ is a measurable function which satisfies the condition $|g(s)| \leq 1/2$ for all $s \in [0, \ell]$. Then

$$z_1(g) + z_2(g) = \int_P g(s)\varphi'(s)\mathrm{d}s + \int_Q g(s)\varphi'(s)\mathrm{d}s = \int_0^\ell g(s)\varphi'(s)\mathrm{d}s,$$

and therefore $Z = Z_1 + Z_2$. We are now going to prove that Z_1 is a convex body and $b(Z_1) \leq h$.

On this purpose we put

$$x(g) = \int_{P_1} g(s)\varphi'(s_1)\mathrm{d}s + \ldots + \int_{P_k} g(s)\varphi'(s_k)\mathrm{d}s.$$

Then

$$x(g) = \varphi'(s_1)\int_{P_1} g(s)\mathrm{d}s + \ldots + \varphi'(s_k)\int_{P_k} g(s)\mathrm{d}s =$$

$$= \frac{1}{\lambda_1}e_1 \int_{P_1} g(s)\mathrm{d}s + \ldots + \frac{1}{\lambda_k}e_k \int_{P_k} g(s)\mathrm{d}s = \nu_1 e_1 + \ldots + \nu_k e_k,$$

where

$$|\nu_i| = \frac{1}{\lambda_i}\left|\int_{P_i} g(s)\mathrm{d}s\right| \leq \frac{1}{\lambda_i}\cdot\frac{1}{2}\operatorname{mes} P_i = \frac{1}{\lambda_i}\cdot\frac{1}{2}q\lambda_i = \frac{1}{2}q.$$

It follows from this that the set M^* of all points $x(g)$ is a zonotope, which is homothetic to M with the multiple q.

Now we have

$$|z_1(g) - x(g)| = \left|\sum_i \left(\int_{P_i} g(s)\varphi'(s)\mathrm{d}s - \int_{P_i} g(s)\varphi'(s_i)\mathrm{d}s\right)\right| \leq$$

$$\leq \sum_i \int_{P_i} |g(s)|\cdot|\varphi'(s) - \varphi'(s_i)|\mathrm{d}s \leq \sum_i \frac{1}{2}\cdot\frac{\delta}{\lambda_1 + \ldots + \lambda_k}\operatorname{mes} P_i =$$

$$= \sum_i \frac{1}{2}\cdot\frac{\delta}{\lambda_1 + \ldots + \lambda_k}\cdot q\lambda_i = \frac{1}{2}q\delta < q\delta.$$

Thus in the $q\delta$-neighborhood of each point $z_1(g) \in Z_1$ there is a point $x(g) \in M^*$ (namely, the point $x(g)$ with the same function $g(s)$). Conversely, in the $q\delta$-neighborhood of each point $x(g) \in M^*$ there is a point $z_1(g) \in Z_1$. Consequently, $d(Z_1, M^*) < q\delta$.

Let γ be a homothety with the multiple $\frac{1}{q}$, such that $\gamma(M^*) = M$. Then

$$d(\gamma(Z_1), M) = d(\gamma(Z_1), \gamma(M^*)) < \frac{1}{q}\cdot q\delta = \delta.$$

This means, in view of the choice of δ, that $\gamma(Z_1) \supset V$, i.e. $\gamma(Z_1)$ is a convex body, and $b(\gamma(Z_1)) \leq h$. It follows from this that $b(Z_1) = b(\gamma(Z_1)) \leq h$, which proves the Lemma. ∎

Lemma 4. *Each zonoid $Z \subset \mathbb{R}^n$ may be represented as a limit of a convergent sequence of its tangential zonotopes.*

Proof. Let $\varphi(s)$, $0 \leq s \leq \ell$, be a determining vector-function of Z. We choose an arbitrary positive number ε and construct an ε-net $\{e_1, \dots, e_p\}$ on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Let E_i $(i = 1, \dots, p)$ be the set of all points $s \in [0, \ell]$, at which $\varphi'(s)$ exists and satisfies the condition $|\varphi'(s) - e_i| < \varepsilon$. Almost all points of the segment $[0, \ell]$ belong to $E_1 \cup \ldots \cup E_p$. We now put

$$F_1 = E_1; \quad F_i = E_i \setminus (E_1 \cup \ldots \cup E_{i-1}), \qquad i = 2, \dots, p.$$

Then each two of the sets $F_1, \dots, F_p$ are disjoint, and almost all points of $[0, \ell]$ belong to $F_1 \cup \ldots \cup F_p$. We shall suppose that all sets $F_1, \dots, F_p$ are nonvoid (void sets may be omitted).

For each $i = 1, \dots, p$ we fix a point $s_i \in F_i$ and put

$$z(g) = \sum_i \int_{F_i} g(s)\varphi'(s)\mathrm{d}s, \quad x(g) = \sum_i \int_{F_i} g(s)\varphi'(s_i)\mathrm{d}s; \qquad |g(s)| \leq \frac{1}{2}.$$

Then the set of all points $z(g)$ is the given zonoid Z, and the set of all points $x(g)$ is a zonotope $M = I_1 + \ldots + I_p$, where I_i is a segment which is parallel to $\varphi'(s_i)$ and has a length $|I_i| = \operatorname{mes} F_i$. We have

$$|z(g) - x(g)| =$$

$$\left| \sum_i \left(\int_{F_i} g(s)\varphi'(s)\mathrm{d}s - \int_{F_i} g(s)\varphi'(s_i)\mathrm{d}s \right) \right| \leq \sum_i \int_{F_i} |g(s)| \cdot |\varphi'(s) - \varphi'(s_i)|\mathrm{d}s \leq$$

$$\sum_i \int_{F_i} |g(s)| \cdot \left(|\varphi'(s) - e_i| + |e_i - \varphi'(s_i)| \right) \mathrm{d}s \leq \sum_i \int_{F_i} \frac{1}{2} \cdot 2\varepsilon \mathrm{d}s = \int_0^\ell \varepsilon \mathrm{d}s = \varepsilon\ell.$$

This means that $d(Z, M) \leq \varepsilon\ell$. This establishes the Lemma (since $\varepsilon > 0$ is arbitrary small). ∎

Lemma 5. *If a zonoid Z is a limit of a convergent sequence M_1, M_2, ... of its tangential zonotopes, then $b(Z) = \lim_{k \to \infty} b(M_k)$.*

It follows obviously from Lemmas 2 and 3. ∎

Now the proof of the theorem follows immediately from the Lemmas 4, 5 and Martini's result [6]. ∎

Let us make some remarks which are connected with our main theorem. First of all, we give an extension of the notions of a zonotope.

We shall say a convex body is an *unbounded zonotope* if it is a vector sum of one-dimensional convex sets, at least one of which is unbounded. An *unbounded zonoid* is a convex set, which is a limit (in Hausdorff's sense) of a consequence $M_1, M_2, \dots$ of unbounded zonotopes.

Let us recall, that the *characteristic cone* of an unbounded convex set M with a vertex $x_0 \in \operatorname{int} M$ is the union of all rays with the initial points x_0, which are contained in M. The characteristic cone of M is determined up to translation. Let us also recall, that a *nearly conic* convex set is an unbounded convex set which is contained in a r-neighborhood of its characteristic cone for some positive number r. It is clear that each unbounded zonotope (and consequently, each unbounded zonoid) is a nearly conic set. For every nearly conic convex body M, Hadwiger's problem is equivalent to the illumination problem [10], [11].

Let M be a nearly conic convex body, K its characteristic cone, L a direct summand for the plane $\operatorname{aff} K$ and $\pi : \mathbb{R}^n \to L$ the projection along $\operatorname{aff} K$. Then the convex set $N = \pi(M) \subset L$ is bounded, and we have the equality $b(M) = b(\operatorname{cl} N)$ [10], [11]. The set $\operatorname{cl} N$ is determined up to an affine transformation. We shall say that $\operatorname{cl} N$ is a *limit projection* of the nearly conic convex body M. It is easy to prove that a limit projection of an unbounded zonotope is a bounded zonotope, and hence a limit projection of an unbounded zonoid is a bounded zonoid. From our main theorem we now obtain the following result:

Let Z be an unbounded zonoid and $\operatorname{cl} N$ its limit projection. We put $p = \dim \operatorname{cl} N$. If $\operatorname{cl} N$ is a parallelotope then $b(Z) = 2^p$, and if $\operatorname{cl} N$ is not a parallelotope then $b(Z) \leq 3 \cdot 2^{p-2}$.

Let us note that a generalization of the integral definition of zonoids (cf. (1)) leads us to a *larger* class of unbounded convex sets, than unbounded zonoids. Namely, let K be an unbounded curve with a vector equation $x = \varphi(s)$, $s \in \mathbb{R}$, where the function $\varphi(s)$ has a finite variation on each segment $[a, b] \subset \mathbb{R}$ and s is a length (with an appropriate sign). Let us consider all the points

$$x(g) = \int_{-\infty}^{\infty} g(s)\varphi'(s)\mathrm{d}s, \tag{5}$$

where $g(s)$ is a measurable function, such that $|g(s)| \leq 1/2$ for all $s \in \mathbb{R}$ and the integral (5) is convergent. Then the set of all points $x(g)$ is an unbounded convex set, and the class of all such sets is larger than the class of all unbounded zonoids.

For example, let K be a parabola in a plane. Then the set Z of all the points (5) is the convex hull of this parabola, i.e. Z is not nearly conic.

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